

Quantum 3D Tensionless String in Light-cone Gauge

Kenta Murase^{1*}

¹*Department of Physics, Kyoto University, Kitashirakawa, Kyoto 606-8502, Japan*

Abstract

We discuss the quantization of a tensionless closed string in light-cone gauge. It is known that by using a Hamiltonian BRST scheme the tensionless p-branes have no Lorentz anomaly in any space-time dimensions and no anomaly of space-time conformal symmetry in two dimensions. In this paper, we show that tensionless 3d strings in light-cone gauge also have no anomaly of space-time conformal symmetry. We also study the spectrum of a tensionless 3d closed string.

*E-mail: kmurase@gauge.scphys.kyoto-u.ac.jp or kenta1murase2@gmail.com

1 Introduction

It is well known that the critical dimension of bosonic (or supersymmetric) string theory is 26 (or 10) and we can check this fact by many methods, including Light-cone quantization and BRST quantization [1, 2]. Recently, Luca Mezincescu and Paul K. Townsend showed that by using light-cone gauge a consistent critical string theory can be constructed also in three dimensions, since there is no Lorentz anomaly in three dimensions. In fact, in three dimensions the dangerous commutator which breaks Lorentz symmetry, $[\mathcal{J}^{-I}, \mathcal{J}^{-J}]$ ($I, J = 2, \dots, D-1$), vanishes trivially because there is only one transverse direction [3, 4, 5].

$$[\mathcal{J}^{-I}, \mathcal{J}^{-I}] \equiv 0 \quad \text{in 3 dim.} \quad (1.1)$$

Moreover they found that the spectrum of 3d light-cone string includes "anyons", which have non half-integer spins.

The difference between the critical dimension of the light-cone gauge quantization and that of others might mean the fault of the light-cone quantization or the incompleteness of other quantizations including BRST method. Mezincescu and Townsend suggests that such difference may be caused by the existence of anyon, although it is still not clear whether this is in the case and how the difference arises. Beside finding a reason for the difference, it is also important to find other examples which give a different result in the light-cone method and others and to invent other quantization schemes which reproduce results obtained by the light-cone method, especially by the covariant one.

In this paper, we investigate 3d tensionless bosonic closed string. It has been known by using BRST method that tensionless p-branes have no Lorentz anomaly in any dimensions and that conformal tensionless p-branes have the critical dimension $D = 2$ [6, 7, 8, 9]. And the mass spectrum also has been investigated [10, 11]. On the other hand, the space-time conformal symmetry has some anomaly in light-cone gauge [12, 13]. We can readily show that no anomalous terms appear in three dimensions for commutators introduced in [12, 13]. However, in order to conclude the absence of space-time conformal anomaly, we need to calculate other nontrivial commutators. In the calculation, we need to introduce some regularization because many troublesome divergences come from non-locality of operators in light-cone gauge. In this paper, removing higher modes of operators than a given cutoff, we find that its nontrivial commutator has no anomalous terms. We also investigate the spectrum of a 3d tensionless bosonic closed string and discuss them.

The content of the paper is as follows: In Section 2, we represent a 3D tensionless closed bosonic string in light-cone gauge. In section 3, we quantize a 3D tensionless closed bosonic string in light-cone gauge and find that this string theory has the space-time conformal symmetry. In Section 4, we discuss the spectrum of 3d tensionless string. In Section 5, we end the paper with the conclusion and outlooks. The definition of light-cone coordinate and the algebra of conformal group are collected in appendices.

2 3D tensionless string in light-cone gauge

In this section we consider in three dimensions the light-cone quantization of a string without tension, namely a tensionless string. We follow the method of [3] to quantize a tensionless string.

First we consider a string with tension, namely a tensile string. A 3D bosonic closed string with tension T is described by Nambu-Goto action:

$$S[\mathbf{X}] = -T \int d\tau \oint \frac{d\sigma}{2\pi} \sqrt{\left(\left(\dot{\mathbf{X}} \cdot \mathbf{X}' \right)^2 - \dot{\mathbf{X}}^2 (\mathbf{X}')^2 \right)}, \quad (2.1)$$

where $\mathbf{X}^\mu(\tau, \sigma); \mu = 0, 1, 2$ represents an embedding of the world sheet (τ, σ) to 3D Minkowski space with matrix $\eta = \text{diag}(-1, 1, 1)$. An overdot indicates a derivative with respect to time parameter τ and

a prime indicates a derivative with respect to string coordinate σ . The centerdot or superscript "2" indicate the contraction. Moreover we assume that the functions are periodic, $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$.

By using the conjugate momentums \mathbf{P}_μ , and the auxiliary fields V and U , we can rewrite the action (2.1) to the form

$$S[\mathbf{X}, \mathbf{P}; V, U] = \int d\tau \oint \frac{d\sigma}{2\pi} \left\{ \dot{\mathbf{X}}^\mu \mathbf{P}_\mu - \frac{1}{2} V [\mathbf{P}^2 + (T\mathbf{X}')^2] - U \mathbf{X}'^\mu \mathbf{P}_\mu \right\}, \quad (2.2)$$

where V and U are the Lagrange multipliers for the Hamiltonian and S^1 -diffeomorphism constraints, respectively. When one eliminates \mathbf{P}_μ , followed by the elimination of V and U in order, the Nambu-Goto action (2.1) is reproduced¹.

The new action (2.2) has a local symmetry under the transformations

$$\begin{aligned} \delta \mathbf{X}^\mu &= \alpha \mathbf{P}^\mu + \beta \mathbf{X}'^\mu, \\ \delta \mathbf{P}_\mu &= T^2 (\alpha \mathbf{X}'_\mu)' + (\beta \mathbf{P}_\mu)', \\ \delta V &= \dot{\alpha} + U' \alpha - U \alpha' + V' \beta - V \beta', \\ \delta U &= \dot{\beta} + U' \beta - U \beta' + T^2 (\alpha V' - \alpha' V), \end{aligned} \quad (2.3)$$

where $\alpha(\tau, \sigma)$ and $\beta(\tau, \sigma)$ are arbitrary functions. We choose light-cone gauge to fix this local symmetry and investigate Lorentz anomaly. We will find that "anyons" appear in the spectrum [3].

From now on, we set T to be zero to investigate the tensionless string. The action (2.2) with $T = 0$ is

$$S[\mathbf{X}, \mathbf{P}; V, U] = \int d\tau \oint \frac{d\sigma}{2\pi} \left\{ \dot{\mathbf{X}}^\mu \mathbf{P}_\mu - \frac{1}{2} V \mathbf{P}^2 - U \mathbf{X}'^\mu \mathbf{P}_\mu \right\}, \quad (2.4)$$

and the gauge symmetry (2.3) becomes

$$\begin{aligned} \delta \mathbf{X}^\mu &= \alpha \mathbf{P}^\mu + \beta \mathbf{X}'^\mu, \\ \delta \mathbf{P}_\mu &= (\beta \mathbf{P}_\mu)', \\ \delta V &= \dot{\alpha} + U' \alpha - U \alpha' + V' \beta - V \beta', \\ \delta U &= \dot{\beta} + U' \beta - U \beta'. \end{aligned} \quad (2.5)$$

In the next subsection we fix the gauge symmetry (2.6) with light-cone gauge.

2.1 Light-cone gauge

The light-cone components of coordinates and their conjugates, (X^+, X^-, X) and (P_+, P_-, P) , are written with the components in Minkowski base as follows:

$$\begin{aligned} X^\pm &\equiv \frac{1}{\sqrt{2}} (\mathbf{X}^1 \pm \mathbf{X}^0), \quad X \equiv \mathbf{X}^2, \\ P_\pm &\equiv \frac{1}{\sqrt{2}} (\mathbf{P}_1 \pm \mathbf{P}_0) = P^\mp, \quad P \equiv \mathbf{P}_2. \end{aligned} \quad (2.6)$$

We impose light-cone gauge to fix the gauge symmetry (2.5),

$$X^+ = \tau, \quad P_- = p_-(\tau) \neq 0, \quad (2.7)$$

¹The action of a relativistic massive point particle is $S = -m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}_\mu}$. This can be written as $S = \int d\tau [\dot{x}^\mu p_\mu - \frac{1}{2} v (p^2 + m^2)]$, where p_μ is conjugate to x^μ and v is the Lagrange multiplier for the off-shell condition, $p^2 + m^2 = 0$. The former action is reproduced by the eliminating p_μ and v in order.

where $p_-(\tau)$ is a non-vanishing function of τ . This gauge choice restricts gauge parameters such that $\alpha = 0, \beta = \beta_0(\tau)$ and then leaves only the residual global gauge symmetry induced by a constant shift of σ . We leave this for a moment to clarify what the constraint is, though we will fix this later.

To obtain the action in light-cone gauge, we decouple the center of mass coordinate which is the average about σ from the rest. Namely, for a given function $F(\tau, \sigma)$, we decompose it into f and \bar{F} :

$$\begin{aligned} f(\tau) &\equiv \oint \frac{d\sigma}{2\pi} F(\tau, \sigma), \\ \bar{F}(\tau, \sigma) &\equiv F(\tau, \sigma) - f(\tau). \end{aligned} \quad (2.8)$$

Note that $\oint \frac{d\sigma}{2\pi} \bar{F} = 0$.

Using the gauge choice (2.7) and the decoupling with regard to $F = X^-, X, P_+, P, U$, we find that the Lagrangian (2.4) reduces to

$$\begin{aligned} L = & \dot{x}p + \dot{x}^- p_- + p_+ + \oint \frac{d\sigma}{2\pi} \dot{X} \bar{P} - u \oint \frac{d\sigma}{2\pi} \bar{X}' P - \oint \frac{d\sigma}{2\pi} \bar{U} \bar{X}' P \\ & + p_- \oint \frac{d\sigma}{2\pi} \left\{ \bar{X}^- \bar{U}' - V \left(P_+ + \frac{1}{2p_-} P^2 \right) \right\}, \end{aligned} \quad (2.9)$$

where \bar{X}^- is a lagrange multiplier giving the constraint $\bar{U}' = 0$. Together with $\oint \frac{d\sigma}{2\pi} \bar{U} = 0$, we obtain $\bar{U} = 0$. On the contrary the variation of \bar{U} induces the relation

$$p_-(\bar{X}^-)' = -\bar{X}' P + \oint \frac{d\sigma}{2\pi} \bar{X}' \bar{P}. \quad (2.10)$$

which we use to determine \bar{X}^- .

Moreover the variation of V leads to

$$P_+ = -\frac{1}{2p_-} P^2. \quad (2.11)$$

We regard this equation as expressing P_+ in terms of other variables. The center part of P_+ is the Hamiltonian

$$H \equiv -p_+ = \frac{1}{2p_-} (p^2 + \mathcal{M}^2), \quad (2.12)$$

and the mass squared is given by

$$\mathcal{M}^2 = 2p_+ p_- - p^2 = \oint \frac{d\sigma}{2\pi} \bar{P}^2. \quad (2.13)$$

In summary the Lagrangian reduces to

$$L = \dot{x}p + \dot{x}^- p_- + p_+ + \oint \frac{d\sigma}{2\pi} \dot{X} \bar{P} - H - u \oint \frac{d\sigma}{2\pi} \bar{X}' P. \quad (2.14)$$

Next we use the residual gauge symmetry induced by β_0 to fix $u = 0$ and rewrite the Lagrangian to the form

$$L = \dot{x}p + \dot{x}^- p_- + \oint \frac{d\sigma}{2\pi} \dot{X} \bar{P} - H \quad (2.15)$$

with a constraint

$$\oint \frac{d\sigma}{2\pi} \bar{X}' P = 0. \quad (2.16)$$

Finally we solve most of the infinite constrains in the action (2.4) by eq. (2.10) and (2.11) and then leave the only onstraint (2.16). It is an advantage of using light-cone gauge that we can solve most of the constraints leaving a finite number of simple constarints. The difference between the light-cone quantization and others originates in whether the number of constraints is finite or infinite.

Fourier expansion

To solve eq. (2.10) and (2.11) explicitly, we use the Fourier expansion of X and P with respect to σ :

$$\begin{aligned} X &= \sum_{n=-\infty}^{\infty} X_n e^{in\sigma}, \quad X_0 = x, \\ P &= \sum_{n=-\infty}^{\infty} P_n e^{in\sigma}, \quad P_0 = p. \end{aligned} \quad (2.17)$$

The reality conditions of X and P lead to

$$\begin{aligned} (X_n)^* &= X_{-n}, \\ (P_n)^* &= P_{-n}, \end{aligned} \quad (2.18)$$

where an asterisk represents the complex conjugate. From now on the sum without an explicit range specified must be understood to run from minus infinity to infinity ². For a tensile string, we usually combine X_n and P_n as

$$\begin{aligned} \alpha_n &= -i\sqrt{\frac{T}{2}}nX_n + \frac{1}{\sqrt{2T}}P_n, \\ \tilde{\alpha}_{-n} &= i\sqrt{\frac{T}{2}}nX_n + \frac{1}{\sqrt{2T}}P_n \end{aligned} \quad (2.19)$$

which express the right-moving or the left-moving respectively. However, because we have no scale like T , it is not clear whether we should introduce some scale to combine Fourier coefficient (2.17) in the oscillator form.

Let us rewrite the mass squared (2.12) and the constraint (2.16). First we solve eq. (2.10) as follows ³:

$$\bar{X}^- = -\frac{1}{p_-} \sum_{n \neq 0} \frac{i}{n} M_n e^{in\sigma}, \quad (2.20)$$

where

$$M_n \equiv -i \sum_m m X_m P_{n-m}, \quad \text{for } n \neq 0. \quad (2.21)$$

Denoting the center part by x^- , we have $X^- = x^- + \bar{X}^-$.

Next we solve eq. (2.11)

$$P_+ = -\frac{1}{2p_-} \sum_n L_n e^{in\sigma}, \quad (2.22)$$

where

$$L_n \equiv \sum_m P_m P_{n-m}. \quad (2.23)$$

From the zero mode L_0 , the mass squared reads as

$$\mathcal{M}^2 = 2 \sum_{n>0} P_n P_{-n}. \quad (2.24)$$

²For example, $\sum_n \equiv \sum_{n=-\infty}^{\infty}$, $\sum_{n \neq 0} \equiv \sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty}$, $\sum_{n>0} \equiv \sum_{n=1}^{\infty}$ and so on.

³The condition $\oint \frac{d\sigma}{2\pi} \bar{X}^- = 0$ leads to the same constraint as (2.16) or (2.25).

Further the constraint (2.16) is expressed as

$$0 = \oint \frac{d\sigma}{2\pi} \bar{X}' P = i \sum_n n X_n P_{-n} \equiv -M_0, \quad (2.25)$$

which corresponds to the level-matching condition for a tensile string.

Equations of motion

Using the Fourier expansion, the Lagrangian (2.15) is written in the form

$$L = \dot{x} p + \dot{x}^- p_- + \sum_{n \neq 0} \dot{X}_n P_{-n} - H, \quad (2.26)$$

where $H = \frac{1}{2p_-}(p^2 + \mathcal{M}^2)$ and the mass squared is given by eq. (2.25). We obtain the equations of motion from this Lagrangian,

$$\dot{p} = \dot{p}_- = 0, \quad \dot{x} = \frac{p}{p_-}, \quad \dot{x}^- = -\frac{H}{p_-} \quad (2.27)$$

and

$$\dot{X}_n = \frac{P_n}{p_-}, \quad \dot{P}_n = 0. \quad (2.28)$$

These equations indicate that the center part and the rest part move linearly with uniform acceleration,

$$X(\tau, \sigma) = X(\tau = 0, \sigma) + \frac{P(\sigma)}{p_-} \tau, \quad (2.29)$$

which is expected from the equations of motion before taking the Fourier expansion. It reflects the fact that the Hamiltonian and the mass squared are independent of τ .

3 Quantization of 3D tensionless string and space-time conformal symmetry

In this section we first quantize a 3D tensionless string represented in section 2 and next find that this theory has no anomaly for the space-time conformal symmetry.

3.1 Quantization

In the quantum theory, the canonical variables in the action (2.15) are promoted to operators with the commutation relations

$$[x^-, p_-] = i, \quad [X(\sigma), P(\sigma')] = 2\pi i \delta(\sigma - \sigma') \quad \text{with others vanishing}, \quad (3.1)$$

where we set $\hbar = 1$. In the mode expansion, the last relation indicates

$$[x, p] = i, \quad [X_n, P_m] = i \delta_{n+m, 0}, \quad (3.2)$$

where $n, m \in \mathbb{Z}$. Using these basic relations, we obtain that L_n and M_n satisfy the following relations:

$$\begin{aligned} [X_n, M_m] &= n X_{n+m}, \quad [P_n, M_m] = n P_{n+m}, \quad [X_n, L_m] = 2n P_{n+m}, \quad [P_n, L_m] = 0 \\ [M_n, M_m] &= (n - m) M_{n+m}, \quad [L_n, M_m] = (n - m) L_{n+m}, \quad [L_n, L_m] = 0, \end{aligned} \quad (3.3)$$

where we use the operator-ordering given in eq. (2.21) and (2.23) . We make a remark that L_n and M_n satisfy the 2-dimensional Galilean Conformal Algebra (2d GCA) ⁴.

The quantum Hamiltonian and the mass squared are then

$$\begin{aligned} H &= \frac{1}{2p_-}(p^2 + \mathcal{M}^2), \\ \mathcal{M}^2 &= 2 \sum_{n>0} P_n P_{-n}, \end{aligned} \tag{3.4}$$

where there is no constant term arising from ambiguity of the operator ordering because P_n commute with P_{-n} ⁵. On the contrary, the constraint (2.25) has ambiguity of the operator ordering. This ambiguity is related to the choice of the vacuum. Here we define M_0 as in (2.25) in order that the action of M_0 on physical state vanishes.

3.2 Generators

In light-cone gauge quantization, the space-time Lorentz and conformal symmetries are not clear. Therefore we check whether the generators of these symmetries satisfy expected commutation relations (B.7) and (B.11), respectively. In $D > 3$, we determine the critical dimension of a bosonic string and the ordering constant in the mass squared to preserve the Lorentz invariance in quantum theory. In three dimensions, there is only one transverse direction which means that the dangerous commutator (1.1) vanishes trivially. Hence we have no Lorentz anomaly. However the conformal symmetry is not trivial even in three dimensions. We now investigate whether a 3D tensionless closed string has conformal symmetry.

The conformal group is generated by translations, Lorentz rotations, dilatation and special conformal transformations. Now we define these generators in the "reference order," which all P_n and p_- are to the right of X_n and x^- respectively [12]. We shall call the reference order "R-order" ⁶. The definitions of L_n and M_n in eq. (2.21) and (2.23) were already into R-order. Hereafter, when we want to emphasize operators to be into R-order, we specify the R-ordered operator by the subscript R .

We first define the translation generators as

$$\mathcal{P}_R^\mu \equiv \oint \frac{d\sigma}{2\pi} P^\mu. \tag{3.5}$$

In the light-cone base, these are

$$\mathcal{P}_R = p, \quad \mathcal{P}_R^+ = p_-, \quad \mathcal{P}_R^- = p_+ = -H. \tag{3.6}$$

Second, Lorentz generators are defined as

$$\mathcal{J}_R^\mu \equiv \epsilon^{\mu\nu\rho} \oint \frac{d\sigma}{2\pi} X_\nu P_\rho. \tag{3.7}$$

⁴Recently this algebra was investigated in terms of tensionless string [14]

⁵In the case of a tensile string, the mass squared has a constant a arising from operator ordering ambiguity. To avoid the Lorentz anomaly, we choose the critical dimension of the string theory and the ordering constant to be $D = 26$ and $a = 1$, respectively. However in three dimensions no Lorentz anomaly exists trivially and then a leave as an arbitrary constant [3].

⁶We can obtain the R-ordering from the normal ordering in the tensionless limit $T \rightarrow 0$. In detail, the string ground state $|0\rangle_T$ of a tensile string with a tension T is annihilated by positive modes of right-moving and left-moving oscillators, $\{\alpha_n, \tilde{\alpha}_n; n > 0\}$. According to eq. (2.19), this string ground state in the tensionless limit reduces the vacuum which annihilates all P_n for all non-zero n . However, when we set $T = 0$ from the beginning, there is no reason why this state should be chosen.

In light-cone base, these are written as

$$\begin{aligned}\mathcal{J}_R &= x^- p_- + \tau H, & \mathcal{J}_R^+ &= \tau p - x p_-, \\ \mathcal{J}_R^- &= -x^- p - x H + \frac{\Lambda}{p_-},\end{aligned}\tag{3.8}$$

where

$$\Lambda = p_- \oint \frac{d\sigma}{2\pi} [\bar{X} \bar{P}_+ - \bar{X}^- \bar{P}] = \sum_{n \neq 0} \left(-\frac{1}{2} X_n L_{-n} + \frac{i}{n} M_n P_{-n} \right).\tag{3.9}$$

Next the dilatation generator is defined as

$$\mathcal{D}_R = \oint \frac{d\sigma}{2\pi} X^\mu P_\mu,\tag{3.10}$$

and now expressed as

$$\mathcal{D}_R = x^- p_- - \tau H + \sum_n X_n P_{-n}.\tag{3.11}$$

At last the generators of the special conformal transformations are defined as

$$\mathcal{K}_R^\mu = \oint \frac{d\sigma}{2\pi} \left[X^\mu (X \cdot P) - \frac{1}{2} (X \cdot X) P^\mu \right]_R,\tag{3.12}$$

where the subscript R in the right-hand side indicates the reordering into R-order⁷. In light-cone base,

$$\begin{aligned}\mathcal{K}_R^+ &= -\frac{1}{2} \sum_n X_n X_{-n} p_- + \tau \sum_n X_n P_{-n} - \tau^2 H \\ \mathcal{K}_R &= x x^- p_- + \sum_{n \neq 0} \frac{i}{n} X_n M_{-n} + \frac{1}{2} \sum_n \sum_m X_n X_m P_{-n-m} + \tau \mathcal{J}^- \\ \mathcal{K}_R^- &= x^- x^- p_- + x^- \sum_n X_n P_{-n} - \frac{i}{p_-} \sum_n \sum_{m \neq 0} \left(\frac{n}{m^2} + \frac{1}{m} \right) X_n M_m P_{-n-m} \\ &\quad + \frac{1}{4p_-} \sum_n \sum_m X_n X_m L_{-n-m}.\end{aligned}\tag{3.13}$$

Physical observables should be represented by Hermitean operators. We require the conformal generators to be Hermitean. In our case the R-ordered definitions of the generators in (3.8)(3.11)(3.13) are simple but not Hermitean⁸. Then we introduce the Hermitean version \mathcal{G} of the R-ordered operator \mathcal{G}_R as follows:

$$\mathcal{G} \equiv \frac{1}{2} \left(\mathcal{G}_R + (\mathcal{G}_R)^\dagger \right).\tag{3.14}$$

All Hermitean versions of generators are independent of τ . Hence, when we deal with Hermitean generators, we can use generators to be set $\tau = 0$. Here note that the differences of the Hermitean version from the R-ordered generators mostly include terms proportional to the infinite number. This indicates the necessity of the regularization.

⁷Because \mathcal{K}_R^- includes a quadratic term of X^- , the simply defined \mathcal{K}^- is not into R-order as it is.

⁸Of course the translation $\mathcal{P} = p$, the Hamiltonian H and the mass squared \mathcal{M}^2 are clearly Hermitean. Moreover Λ is Hermitean, and if $\sum_n n = 0$ the constraint M_0 is also Hermitean.

3.3 Anomaly

In the previous subsection, we defined the generators of the space-time conformal symmetry. In our case, there is the only one transverse direction and the dangerous commutator (1.1) vanishes trivially. Therefore all commutation relations of Poincaré group (B.7) are satisfied. In this subsection, we investigate whether the anomaly arises in the quantization of a 3D tensionless string, namely whether all commutation relations of conformal symmetry (B.12) are satisfied.

In [12], J. Isberg et. al. showed that in $D > 3$ the anomaly arises from the commutator $[\mathcal{K}^I, \mathcal{J}^{-J}]$ regardless of the ordering. The right hand side of this commutator has the off-diagonal, traceless part with respect to I, J as well as the trace part. The difference of trace part can be absorbed in the redefinition of \mathcal{K}^- , but the traceless part remains as the anomaly. In three dimensions, this commutator is only one and hence this type of anomaly does not exist. But we need to calculate other nontrivial commutators to check the space-time conformal symmetry. After the tedious calculation, we will find that many of commutators are as expected but a number of commutators not.

One of "bad" commutators is $[\mathcal{K}, \mathcal{J}^-]$ which corresponds to $[\mathcal{K}^I, \mathcal{J}^{-J}]$ in $D > 3$. Though this is slightly different from \mathcal{K}^- , we can interpret that this give the redefinition as

$$\hat{\mathcal{K}}^- = i[\mathcal{K}, \mathcal{J}^-]. \quad (3.15)$$

Note that in the term with M_0 , we put M_0 to the right of other operators and set zero.

Another one of "bad" commutators is $[\hat{\mathcal{K}}^-, \mathcal{J}^-]$, which must be zero:

$$[\hat{\mathcal{K}}^-, \mathcal{J}^-] = 0. \quad (3.16)$$

This commutator has quintic terms with respect to X_n and P_n and complicated divergences too. The calculation of this commutator is very laborious and requires a care. Though the calculations of other commutators also have divergences, we can deal with them without the concrete regularization. However, those commutators have many types of divergences⁹ and it is too complicated to calculate them correctly. Moreover we should take care of the shift of dummy variables in the sum and the termwise re-summation.¹⁰ Thus we need the regularization.

Using the redefined $\hat{\mathcal{K}}^-$, $[\hat{\mathcal{K}}^-, \mathcal{J}^-] = 0$ and Jacobi identities,¹¹ the rest of commutators are calculated straightforwardly.

3.4 Cutoff regularization

We use the cutoff regularization to remove higher modes of X and P ,

$$X_n = P_n = 0 \quad \text{for } |n| > N, \quad (3.17)$$

where we assume that N is a large integer. Using this, we obtain

$$M_n = -i \sum_{|m| \leq N, |n-m| \leq N} m X_m P_{n-m}, \quad L_n = \sum_{|m| \leq N, |n-m| \leq N} P_m P_{n-m}, \quad (3.18)$$

and also find

$$M_n = L_n = 0 \quad \text{for } |n| > 2N. \quad (3.19)$$

⁹For example, $\sum_n 1$ and $\sum_n n$.

¹⁰For operators into some order, $\sum_n X_n P_{-n}|_* = \sum_n X_{n+k} P_{-n-k}|_*$, where the subscripts mean these terms are into some order. But for the number, $\sum_n n \neq \sum_n (n+k) \neq \sum_n n+k \sum_n 1$.

¹¹Jacobi identities must be discussed. L_n and M_n into R-order and the simple Hermitean versions of them, as well as X_n and P_n , satisfy Jacobi identities. Namely, when we use the R-ordering or its simple Hermitean version, Jacobi identities do not cause problems under the cutoff regularization.

The range of summation in generators changes as $\sum_n \rightarrow \sum_{|n| \leq N}$.

Thanks to the cutoff regularization, the summations reduce to finite sum and the termwise treatments are possible. And the error as to the shift of dummy variables will decrease. Further, because of symmetric cutoff with respect to positive and negative modes, $\sum_n n = 0$. Thus we resolve many difficulties for the divergence. But we have more terms, due to the informations for the range of the summation.

Though we have only to do the lengthy calculation.

3.5 Check of anomaly free

Our goal is to check a relation

$$[\hat{\mathcal{K}}^-, \mathcal{J}^-] = 0 \quad (3.20)$$

under the cutoff regularization. We can check that other commutation relations are satisfied without using the cutoff regularization.

We summary steps of the calculation, instead of representing the process of the lengthy calculation explicitly. If we use Hermitean operators from the beginning, we have about twice terms than in the case of R-ordered operators. Then we first use R-ordered operators ¹² and next consider their Hermitean versions.

First we calculate

$$[\hat{\mathcal{K}}_R^-, \mathcal{J}^-], \quad (3.21)$$

and order the result into R-order. Here note that

$$\mathcal{J}^- = \mathcal{J}_R^- + \frac{1}{2}i\frac{p}{p_-}. \quad (3.22)$$

Next we consider the anti-Hermitean version of eq. (3.21),

$$[\hat{\mathcal{K}}^-, \mathcal{J}^-] = \frac{1}{2} \left([\hat{\mathcal{K}}_R^-, \mathcal{J}^-] - [\hat{\mathcal{K}}_R^-, \mathcal{J}^-]^\dagger \right). \quad (3.23)$$

The left-hand side of this is the commutator (3.20). The calculations of eqs. (3.21) or (3.23) are very lengthy and laborious. The commutators (3.21) or (3.23) have quintic terms with $XXPPP$ -form, cubic terms with XPP -form and linear terms with P -form. Hence we divide the calculation into three steps.

In the first step, we consider the linear terms. The sum of subscripts in each terms is zero and hence the linear terms with P -form must be proportional to $P_0 = p$. We can check the cancellation of the terms with p easily because the degree of all terms with p decreases. The check of this can be done not only in eq. (2.24) but in eq. (3.21). Because there is no term with p , there is also no linear term.

In the second step, we consider the cubic terms. In the anti-Hermitean version (3.23), we can deform all cubic terms to the form with $[X, P]$ ¹³. Therefore all cubic terms reduce to the linear term and vanish in the first step.

In the third step, we consider the quintic terms. The quintic terms can not be deformed the forms with a commutator $[X, P]$ unlike the case of the cubic terms. Hence the calculation is more difficult

¹²R-ordered generators do not satisfy eq. (3.20)

¹³For example, for $\sum_{|n| \leq N} \sum_{0 < |m| \leq N} \frac{n}{m} X_n P_m P_{-n-m}$, its anti-Hermitean version is $\frac{1}{2} \sum_{|n| \leq N} \sum_{0 < |m| \leq N} \frac{n}{m} (X_n P_m P_{-n-m} - P_{n+m} P_{-m} X_{-n}) = \frac{1}{2} \sum_{|n| \leq N} \sum_{0 < |m| \leq N} \frac{n}{m} (X_n P_m P_{-n-m} - P_{-n-m} P_m X_n) = \frac{1}{2} \sum_{|n| \leq N} \sum_{0 < |m| \leq N} \frac{n}{m} [X_n, P_m] P_{-n-m}$.

than in the other steps. When we are under the cutoff regularization, the calculation is even more difficult. Here note that all summation of quintic terms are the formal sums with respect to operators, not the number. Then we can take a limit $N \rightarrow \infty$ and shift dummy variables and sum term by term naively. After the lengthy calculation which became easy a little, we find the cancellation of quintic terms. On checking of this cancellation, we order quintic terms of eq. (3.21) into R-order and set $M_0 = 0$, where this M_0 is to the right of other operators.

Thus we can check the commutation relation (3.20) and find that there is no anomaly of space-time conformal symmetry in a 3D tensionless bosonic closed string.

4 Spectrum

In this section, we investigate the spectrum of a 3D tensionless closed string. We deal the center of mass part and the rest separately and assume that the ground state of the center is $|p, p_- \rangle$ like the case of a tensile string. We do not care about the ground state of the center part so much.

On the other hand, we need the discussion about the rest part, which includes non-zero modes. First we consider the R-ordered string ground state $|0\rangle_R$, which satisfy

$$P_n|0\rangle_R = 0 \quad \text{for all non-zero } n. \quad (4.1)$$

We can obtain this R-ordered string ground state from the string ground state in tensionless limit [12].

When we choose this string ground state, we obtain other states by acting $\{X_n; n \neq 0\}$ on $|0\rangle_R$. The fundamental elements of states are

$$X_{n_1} X_{n_2} \cdots X_{n_l} |0\rangle_R, \quad (4.2)$$

where n_i ($i = 1, 2, \dots, l$) are non-zero integers. We combine these state to obtain the general states.

Here note that the physical states must satisfy $M_0 = 0$. This means that the physical states with the form (4.2) satisfy

$$\sum_{i=1}^l n_i = 0. \quad (4.3)$$

The general physical states satisfy this condition in each terms. Under this condition, we try to make the eigenstate of \mathcal{M}^2 . In the case of R-ordered string ground state, it is convenient to represent P_n as

$$P_n = -i \frac{\partial}{\partial X_{-n}}, \quad [X_n, P_m] = i \delta_{n+m, 0} \quad (4.4)$$

Using this representation, \mathcal{M}^2 is expressed as

$$\mathcal{M}^2 = -2 \sum_{n>0} \frac{\partial}{\partial X_{-n}} \frac{\partial}{\partial X_n}. \quad (4.5)$$

By space-time conformal symmetry, we expect that eigenvalues of \mathcal{M}^2 are continuous or zero. Because we get the eigenfunction of \mathcal{M}^2 with the eigenvalue $\lambda^{-2} M^2$ by transforming $X_n \rightarrow \lambda X_n$ for all n if we find the eigenfunction with the eigenvalue M^2 . Here we are especially interested in the case of zero eigenvalue, that is massless.

4.1 Mass eigenstate

In this subsection, we use the very similar method in [10]. In order to find eigenvalues of \mathcal{M}^2 and their eigenfunctions, we rewrite eq.(4.5),

$$\mathcal{M}^2 = -\frac{1}{2} \sum_{n>0} \left[\frac{\partial^2}{\partial r_n^2} + \frac{1}{r_n} \frac{\partial}{\partial r_n} + \frac{1}{r_n^2} \frac{\partial^2}{\partial \theta_n^2} \right] \quad (4.6)$$

where r_n and θ_n are real operators that are defined by $X_n = r_n e^{i\theta_n}$ and $X_{-n} = r_n e^{-i\theta_n}$ for all positive n . Because we can decompose the terms of different n , we consider the following differential equation:

$$-\frac{1}{4} \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] \psi_m(r, \theta) = m^2 \psi_m(r, \theta), \quad (4.7)$$

where $r \geq 0$ and $m^2 \geq 0^{14}$. Here we assume $\psi_{m,s}(r, \theta) = \phi_{m,s}(r) e^{is\theta}$, where s is integer and $\phi_{m,s}(r)$ is the function which depends on only r . Then we obtain

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + 4m^2 - \frac{s^2}{r^2} \right] \phi_{m,s}(r) = 0. \quad (4.8)$$

4.1.1 $m^2 > 0$

For $m^2 > 0$, we replace r with $\hat{r} = 2mr$ and obtain

$$\left[\frac{d^2}{d\hat{r}^2} + \frac{1}{\hat{r}} \frac{d}{d\hat{r}} + 1 - \frac{s^2}{\hat{r}^2} \right] \phi_{m,s} \left(\frac{\hat{r}}{2m} \right) = 0, \quad (4.9)$$

where $m > 0$. The solution of this equation is expressed in terms of Bessel function $J_{|s|}(\hat{r})$. In this way, we get the solution $\psi_{m,s}(r, \theta) = N_m J_{|s|}(2mr) e^{is\theta}$, where N_m is the normalization constant¹⁵.

We rewrite the solutions in terms of X_n and X_{-n} with $n > 0$:

$$\begin{aligned} \psi_{m_n, \pm|s_n|}(r_n, \theta_n) &= N_{m_n} J_{|s_n|} (2m_n (X_n X_{-n})^{\frac{1}{2}}) \left(\frac{X_n}{X_{-n}} \right)^{\frac{s}{2}} \\ &= N_{m_n} (m_n X_{\pm n})^{|s|} \sum_{l=0}^{\infty} \frac{(-m_n^2)^l}{l! (l+|s|)!} (X_n X_{-n})^l, \end{aligned} \quad (4.10)$$

where we use

$$J_\nu(z) = \left(\frac{z}{2} \right)^\nu \sum_{l=0}^{\infty} \frac{(-1)^l}{l! \Gamma(l+\nu+1)} \left(\frac{z}{2} \right)^{2l}. \quad (4.11)$$

Thus we find that the eigenfunctions are the combination of positive integer power with respect to X_n and X_{-n} as in eq.(4.2).

Now we consider the normalization of wave functions. Since m^2 is not discrete valuable, the wave functions for each m^2 can not be normalized to "1" but the wave functions for different values of m^2 must be orthogonal. Here we consider the next scalar product. The scalar product of two wave functions $\psi_1(r, \theta)$ and $\psi_2(r, \theta)$ is

$$(\psi_1, \psi_2) = \int_0^\infty dr \int_0^{2\pi} d\theta r \psi_1^*(r, \theta) \psi_2(r, \theta). \quad (4.12)$$

¹⁴Because the eigenvalue of the operator AA^\dagger is zero or positive number, we consider the case of $m^2 \geq 0$. And we also find that the solutions for $m^2 < 0$ have bad behavior in $r \rightarrow \infty$ and can not be normalized.

¹⁵Note that $J_\nu(\hat{r}) = (-1)^\nu J_{-\nu}(\hat{r})$ for integer ν .

Then we obtain for $m > 0$ and $m' > 0$

$$(\psi_{m,s}, \psi_{m',s'}) = \frac{\pi}{2} \frac{|N_m|^2}{m} \delta(m - m') \delta_{s,s'} \quad (4.13)$$

and find the orthogonality. The detail of the normalization is given in appendix.

4.1.2 $m^2 = 0$

For $m^2 = 0$, eq.(4.8) becomes

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{s^2}{r^2} \right] \phi_{0,s}(r) = 0. \quad (4.14)$$

The solutions of this equation for $s \neq 0$ are $r^{|s|}$ and $r^{-|s|}$. For $s = 0$, the solutions are constant and $\log r$. Because of the orthogonality between the eigenfunctions with $m^2 > 0$, $r^{|s|}$ for $s \neq 0$ and constant for $s = 0$ are chosen [10]¹⁶. In terms of X_n and X_{-n} with $n > 0$, the eigenfunctions $\psi_{0,s}(r, \theta)$ are $(X_n)^s$ for $s > 0$, $(X_{-n})^{-s}$ for $s < 0$ and constant for $s = 0$. If we collect these three cases, we get

$$\psi_{0,\pm|s|}(r, \theta) \propto (X_{\pm n})^{|s|}. \quad (4.15)$$

4.1.3 Total eigenfunction

The eigenfunctions of \mathcal{M}^2 are the product

$$\Psi = \prod_{n>0} \psi_{m_n, s_n}(r_n, \theta_n) \quad (4.16)$$

and their eigenvalue are

$$\mathcal{M}^2 = 2 \sum_{n>0} (m_n)^2. \quad (4.17)$$

4.2 Physical spectrum

The physical state must satisfy the constraint $M_0 = 0$. In terms of r_n and θ_n , M_0 becomes

$$M_0 = -i \sum_n n X_n P_{-n} = - \sum_{n>0} n \left[X_n \frac{\partial}{\partial X_n} - X_{-n} \frac{\partial}{\partial X_{-n}} \right] = i \sum_{n>0} n \frac{\partial}{\partial \theta_n}. \quad (4.18)$$

Then the constraint in the case of (4.16) is

$$\sum_{n>0} n s_n = 0. \quad (4.19)$$

As mentioned in the beginning of this section, the eigenvalues of \mathcal{M}^2 are continuous or zero. Though this fact is expected by conformal symmetry, it is understood by the explicit solution (4.10). We get the continuous eigenvalues if $m_n^2 > 0$ at least for some n . On the other hand, we get massless only if $m_n = 0$ for all n . Since we are interested in massless states which are expected to preserve space-time conformal symmetry, we do not discuss the continuous spectrum anymore. Now we investigate the case of massless in detail.

¹⁶The detail is in appendix.

Massless states

If we get the massless states, we have to choose the solution (4.15) for all positive integer n . In other words, we should choose the states (4.2) such that their subscripts satisfy

$$X_{n_1}X_{n_2}\cdots X_{n_l}|0\rangle_R \text{ with } \sum_{i=1}^l n_i = 0 \text{ and } n_i + n_j \neq 0 \text{ for } \forall i, j. \quad (4.20)$$

One of the simplest examples is

$$X_2X_{-1}X_{-1}|0\rangle_R. \quad (4.21)$$

Because a linear term of X_n is prohibited by the constraint $\sum_{i=1}^l n_i = 0$ and the squared terms like X_nX_{-n} do not create massless states.

The theory which is consisted only in massless states are expected to be space-time conformal invariant. Here we define the conformal dimension from the dilatation as

$$\Delta_R \equiv i\mathcal{D}_R. \quad (4.22)$$

We can use this operator to investigate the properties of the massless states. We find that Δ_R count the number of X -type operators by the commutators

$$[X_n, \Delta_R] = X_n. \quad (4.23)$$

For example, we can determine that the state (4.20) has $\Delta_R = l$ and particularly the string ground state $|0\rangle_R$ has $\Delta_R = 0$.

Though we can characterize the theory by using a Casimir operator obtained from the Lorentz subgroup in conformal group, we want to investigate the properties of the massless states by using it in another paper [15].

Finally we comment that the simple Hermitean version of $|0\rangle_R$ and the massless state created from it should be obtained from the massless state. But we do not discuss these things in this paper.

5 Conclusion and Outlook

In this paper, we showed that a 3D tensionless bosonic closed string in light-cone gauge have no anomaly of the space-time conformal symmetry under the cutoff regularization. We here got the different result from that in [12]. Further we investigated the spectrum of 3D tensionless string, particularly massless states. When we consider in R-order, we obtained the simple expression.

Note that the results we got is in the case of single string. We don't understand anything about multi-string theory with interactions, as well as in the case of [3, 4, 5]. It is interesting to investigate the effects of the interaction. To study this in detail, we may need the string field theory.

A 3D tensionless string has some of prospects.

First, one may image the open string version of our results. We can fix the gauge in the same way as our case. In the case of the closed string, we could use the Fourier expansion with respect to σ because of the periodic condition of σ . However, because the equations of motion in a tensionless string are $\ddot{X}^\mu(\sigma, \tau) = 0$, we have no periodic properties as to σ or its combination of τ in the case of a open tensionless string, unlike the tensile case. Hence we can not use the Fourier expansion and need the discussion in another way. For example, one may deal the functions $X(\sigma)$ or $P(\sigma)$ without using the mode expansions. But it may be difficult to think the concrete regularization.

Next, there is a possibility of the other background metric, which we didn't comment on this in this paper. We find the invariance of the action(2.4) under the space-time Weyl transformation with the

transformation of V . Then we expect the possibilities of applications to other background space-times, e.g. AdS_3 , though we must investigate whether this classical fact is satisfied in quantum theory.

Other possibilities are about the supersymmetric version, the membrane version, the relation of Higher-spin gauge theories and so on. In the case of a tensionless closed supersymmetric string, we can use the similar way to ours, unlike the tensionless open string. We expect that the supersymmetric case in light-cone gauge gives the different result from the case of BRST quantization [16, 17]. And it is interesting to consider not only string but also two dimensional object, so-called the membrane. Moreover we are interested in the relation of Higher-spin gauge theories [18, 19], which may be one of the limits of the string theory.

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A Light-cone components

We define the components of light-cone base in this paper.

In D dimensions Cartesian coordinates $(\mathbb{X}^\mu ; \mu = 0, 1, \dots, D-1)$, we define the Minkowski metric $\eta_{\mu\nu} ; \mu, \nu = 0, 1, \dots, D-1$ such that

$$\eta = \text{diag}(-1, 1, \dots, 1). \quad (\text{A.1})$$

The light-cone components of coordinates are

$$X^\pm \equiv \frac{1}{\sqrt{2}} (\mathbb{X}_1 \pm \mathbb{X}_0), \quad X_I \equiv \mathbb{X}_I \quad (I = 2, \dots, D-1). \quad (\text{A.2})$$

Similarly, the light-cone components of an arbitrary vector $(\mathbb{V}_\mu ; \mu = 0, 1, \dots, D-1)$ are

$$V_\pm \equiv \frac{1}{\sqrt{2}} (\mathbb{V}_1 \pm \mathbb{V}_0) = V^\mp, \quad V_I \equiv \mathbb{V}_I \quad (I = 2, \dots, D-1). \quad (\text{A.3})$$

The indices are raised and lowered with metric η . Moreover the inner product is given as

$$-\mathbb{V}_0^2 + \sum_{i=1}^{D-1} \mathbb{V}_i^2 \equiv \mathbb{V}^2 = 2V_+V_- + \sum_{I=2}^{D-1} V_I^2 \quad (\text{A.4})$$

In three dimensions, we emphasize that the transverse direction $(I = 2, \dots, D-1)$ is only one $(I = 2)$. Therefore the two rank anti-symmetric tensor is rewritten as the vector. For example $S^{\mu\nu} = -S^{\nu\mu}$ is represented as

$$S^\mu \equiv \epsilon^{\mu\nu\rho} S_{\nu\rho}, \quad (\text{A.5})$$

where $\epsilon^{\mu\nu\rho}$ is the totally antisymmetric tensor such that $\epsilon^{012} = 1$ in Minkowski base and $\epsilon^{+-2} = 1$ in light-cone base.

B Generators

B.1 Poincaré group

We expect string theory to have the Poincaré symmetry. The quantization induces Lorentz anomaly, and the cancellation of this anomaly determines the critical dimension and the ordering constant. First we consider in generic dimensions D and next restrict ourselves to three dimensions.

The generators of Poincaré group are translations \mathcal{P}_μ , Lorentz rotations $\mathcal{J}^{\mu\nu}$ such that

$$\begin{aligned} [\mathcal{P}^\mu, \mathcal{P}^\nu] &= 0, \quad [\mathcal{P}^\mu, \mathcal{J}^{\rho\sigma}] = i(\eta^{\mu\sigma}\mathcal{P}^\rho - \eta^{\mu\rho}\mathcal{P}^\sigma), \\ [\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] &= i(\eta^{\nu\sigma}\mathcal{J}^{\mu\rho} - \eta^{\nu\rho}\mathcal{J}^{\mu\sigma} - \eta^{\mu\sigma}\mathcal{J}^{\nu\rho} + \eta^{\mu\rho}\mathcal{J}^{\nu\sigma}). \end{aligned} \quad (\text{B.1})$$

In light-cone base, minding $\eta_{+-} = \eta^{+-} = 1$ and $\eta_{IJ} = \delta_{IJ}$, we find the following commutation relations:

$$\begin{aligned} [\mathcal{P}^\pm, \mathcal{J}^{+-}] &= \pm i\mathcal{P}^\pm, \quad [\mathcal{P}^\pm, \mathcal{J}^{\mp I}] = -i\mathcal{P}_I, \\ [\mathcal{P}^I, \mathcal{J}^{\pm J}] &= i\delta^{IJ}\mathcal{P}^\pm, \quad [\mathcal{P}^I, \mathcal{J}^{JK}] = i(\delta^{IK}\mathcal{P}^J - \delta^{IJ}\mathcal{P}^K), \\ [\mathcal{J}^{+-}, \mathcal{J}^{\pm I}] &= \mp i\mathcal{J}^{\pm I}, \quad [\mathcal{J}^{+I}, \mathcal{J}^{-K}] = i(\mathcal{J}^{IK} + \delta^{IK}\mathcal{J}^{+-}), \\ [\mathcal{J}^{\pm I}, \mathcal{J}^{KL}] &= i(\delta^{IL}\mathcal{J}^{\pm K} - \delta^{IK}\mathcal{J}^{\pm L}), \\ [\mathcal{J}^{IJ}, \mathcal{J}^{KL}] &= i(\delta^{JL}\mathcal{J}^{IK} - \delta^{JK}\mathcal{J}^{IL} - \delta^{IL}\mathcal{J}^{JK} + \delta^{IK}\mathcal{J}^{JL}), \end{aligned} \quad (\text{B.2})$$

with others vanishing,

where $I, J, K, L = 2, \dots, D-1$. Among commutators which are zero, particularly

$$[\mathcal{J}^{-I}, \mathcal{J}^{-J}] \stackrel{?}{=} 0 \quad (\text{B.3})$$

cannot be satisfied when we quantize the string theory but if in the special case. This is so called a "dangerous commutator".

3D

In three dimensions Lorentz generators are rewritten as $\mathcal{J}^\pm = \mp \mathcal{J}^{\pm 2}$, $\mathcal{J} = \mathcal{J}^{+-}$ like eq. (A.5). Hence the commutation relations (B.1) become simple as follows,

$$[\mathcal{P}^\mu, \mathcal{P}^\nu] = 0, \quad [\mathcal{J}^\mu, \mathcal{P}^\nu] = i\epsilon^{\mu\nu\rho}\mathcal{P}_\rho, \quad [\mathcal{J}^\mu, \mathcal{J}^\nu] = i\epsilon^{\mu\nu\rho}\mathcal{J}_\rho. \quad (\text{B.4})$$

Moreover we can make two Poincaré Casimir operators easily as follow,

$$M^2 = -\mathcal{P}^2, \quad \Lambda = \mathcal{P}_\mu \mathcal{J}^\mu \quad (\text{B.5})$$

Unitary irreducible representations of Poincaré group are labeled by the value of these two Casimirs[20] and particularly irreps. with $M^2 \geq 0$ are only physical. When $M^2 > 0$ we define relativistic helicity by

$$s = \frac{\Lambda}{M} \quad (\text{B.6})$$

This may take either sign, and parity flips the sign of s . Further we call $|s|$ "spin". If Lorentz group is $SO(1, 2)$, its double cover $SL(2; \mathbb{R})$ or its universal cover, s is an integer, half-integer or any real number.

In light-cone base commutation relations of 3d Poincaré group are

$$\begin{aligned} [\mathcal{J}^\pm, \mathcal{P}^\mp] &= \pm i\mathcal{P}, \quad [\mathcal{J}, \mathcal{P}^\pm] = \pm i\mathcal{P}^\pm, \quad [\mathcal{J}^\pm, \mathcal{P}] = \mp i\mathcal{P}^\pm, \\ [\mathcal{J}, \mathcal{J}^\pm] &= \pm i\mathcal{J}^\pm, \quad [\mathcal{J}^+, \mathcal{J}^-] = i\mathcal{J}, \quad \text{with others vanishing.} \end{aligned} \quad (\text{B.7})$$

In three dimensions the commutator $[\mathcal{J}^-, \mathcal{J}^-]$, corresponding to a dangerous commutator (B.3), vanish trivially because the transverse direction is only one. Therefore 3d string theory has no Lorentz anomaly and is thought of preserving the Poincaré symmetry.

B.2 Conformal group

We expect tensionless string to have space-time conformal symmetry. First we consider in generic dimensions D and next in three dimensions.

The generators of space-time conformal group are dilatation \mathcal{D} and special conformal transformation \mathcal{K}^μ , additional to translations \mathcal{P}_μ and Lorentz rotations $\mathcal{J}^{\mu\nu}$, such that

$$\begin{aligned} [\mathcal{D}, \mathcal{P}^\mu] &= i\mathcal{P}^\mu, \quad [\mathcal{D}, \mathcal{J}^{\mu\nu}] = 0, \quad [\mathcal{D}, \mathcal{K}^\mu] = -i\mathcal{K}^\mu, \quad [\mathcal{K}^\mu, \mathcal{K}^\nu] = 0, \\ [\mathcal{K}^\mu, \mathcal{P}^\nu] &= i(\eta^{\mu\nu}\mathcal{D} + \mathcal{J}^{\mu\nu}), \quad [\mathcal{K}^\mu, \mathcal{J}^{\rho\sigma}] = i(\eta^{\mu\sigma}\mathcal{K}^\rho - \eta^{\mu\rho}\mathcal{K}^\sigma). \end{aligned} \quad (\text{B.8})$$

In light-cone base,

$$\begin{aligned} [\mathcal{D}, \mathcal{P}^\pm] &= i\mathcal{P}^\pm, \quad [\mathcal{D}, \mathcal{P}^I] = i\mathcal{P}^I, \quad [\mathcal{D}, \mathcal{K}^\pm] = -i\mathcal{K}^\pm, \quad [\mathcal{D}, \mathcal{K}^I] = -i\mathcal{K}^I, \\ [\mathcal{K}^\pm, \mathcal{P}^\mp] &= i(\mathcal{D} \pm \mathcal{J}^{+-}), \quad [\mathcal{K}^\pm, \mathcal{P}^I] = -[\mathcal{K}^I, \mathcal{P}^\pm] = i\mathcal{J}^{\pm I}, \\ [\mathcal{K}^I, \mathcal{P}^J] &= i(\delta^{IJ}\mathcal{D} + \mathcal{J}^{IJ}), \\ [\mathcal{K}^\pm, \mathcal{J}^{+-}] &= \pm i\mathcal{K}^\pm, \quad [\mathcal{K}^\pm, \mathcal{J}^{\mp I}] = -i\mathcal{K}^I, \quad [\mathcal{K}^I, \mathcal{J}^{\pm J}] = -i\delta^{IJ}\mathcal{K}^\pm, \\ [\mathcal{K}^I, \mathcal{J}^{JK}] &= i(\delta^{IL}\mathcal{K}^K - \delta^{IK}\mathcal{K}^L), \quad \text{with others vanishing.} \end{aligned} \quad (\text{B.9})$$

Tensionless string in generic dimensions can have a dangerous commutator as well as the case of Lorentz anomaly. That is as follows:

$$[\mathcal{K}^I, \mathcal{J}^{-J}] \stackrel{?}{=} -i\delta^{IJ}\mathcal{K}^-. \quad (\text{B.10})$$

The right hand side of this commutator has the off-diagonal, traceless part with respect to I, J as well as the trace part. The difference of trace part can be absorbed in the redefinition of \mathcal{K}^- . However the traceless part remain as anomaly [12].

3D

In three dimensions the commutation relations (B.8) become simple a little as follows,

$$\begin{aligned} [\mathcal{D}, \mathcal{P}^\mu] &= i\mathcal{P}^\mu, \quad [\mathcal{D}, \mathcal{J}^\mu] = 0, \quad [\mathcal{D}, \mathcal{K}^\mu] = -i\mathcal{K}^\mu, \quad [\mathcal{K}^\mu, \mathcal{K}^\nu] = 0, \\ [\mathcal{K}^\mu, \mathcal{P}^\nu] &= i(\eta^{\mu\nu}\mathcal{D} - \epsilon^{\mu\nu\rho}\mathcal{J}_\rho), \quad [\mathcal{K}^\mu, \mathcal{J}^\nu] = i\epsilon^{\mu\nu\rho}\mathcal{K}_\rho. \end{aligned} \quad (\text{B.11})$$

In light-cone base,

$$\begin{aligned} [\mathcal{D}, \mathcal{P}^\pm] &= i\mathcal{P}^\pm, \quad [\mathcal{D}, \mathcal{P}] = i\mathcal{P}, \quad [\mathcal{D}, \mathcal{K}^\pm] = -i\mathcal{K}^\pm, \quad [\mathcal{D}, \mathcal{K}] = -i\mathcal{K} \\ [\mathcal{K}^\pm, \mathcal{P}^\mp] &= i(\mathcal{D} \pm \mathcal{J}), \quad [\mathcal{K}^\pm, \mathcal{P}] = -[\mathcal{K}, \mathcal{P}^\pm] = \pm i\mathcal{J}^\pm, \quad [\mathcal{K}, \mathcal{P}] = i\mathcal{D}, \\ [\mathcal{K}^\pm, \mathcal{J}^\mp] &= \pm i\mathcal{K}, \quad [\mathcal{K}^\pm, \mathcal{J}] = \mp i\mathcal{K}^\pm, \quad [\mathcal{K}, \mathcal{J}^\pm] = \pm i\mathcal{K}^\pm, \\ &\text{with others vanishing.} \end{aligned} \quad (\text{B.12})$$

In three dimensions the commutator $[\mathcal{K}, \mathcal{J}^-]$, corresponding to a dangerous commutator (B.10), is only one commutator and is regarded as the redefinition of \mathcal{K}^- . Therefore this type of anomaly does not exist. However there are many commutators not to be familiar with tensile string theory and we must check that they satisfy the expected commutation relation of conformal group. The calculation of them is very complicated and lengthy, in particular the next commutation relation is very troublesome;

$$[\mathcal{K}^-, \mathcal{J}^-] = 0. \quad (\text{B.13})$$

A check of this relation is the main result of this paper.

C Normalization of the wave functions

In section 4 we have investigated the eigenfunctions of mass squared operator \mathcal{M}^2 . In this section we give the detail with respect to the normalization of the wave functions [10].

First we check eq.(4.13). By the definition (4.12), the scalar product of $\psi_{m,s}$ and $\psi_{m',s'}$ is

$$(\psi_{m,s}, \psi_{m',s'}) = 2\pi\delta_{s,s'} N_m^* N_{m'} \int_0^\infty dr r J_s(2mr) J_s(2m'r). \quad (\text{C.1})$$

The integral in (C.1) can be computed by using the following result

$$\int_0^y dx x J_l(ax) J_l(bx) = \frac{y}{a^2 - b^2} [a J_{l+1}(ay) J_l(by) - b J_l(ay) J_{l+1}(by)], \quad (\text{C.2})$$

where a and b are positive. We take the limit $y = \Lambda \rightarrow \infty$ of eq.(C.2) and obtain

$$\begin{aligned} \int_0^\infty dx x J_l(ax) J_l(bx) &= \lim_{\Lambda \rightarrow \infty} \frac{\Lambda}{a^2 - b^2} [a J_{l+1}(a\Lambda) J_l(b\Lambda) - b J_l(a\Lambda) J_{l+1}(b\Lambda)] \\ &= \frac{1}{\pi} \frac{1}{\sqrt{ab}} \lim_{\Lambda \rightarrow \infty} \left[\frac{\sin(a-b)\Lambda}{a-b} - (-1)^l \frac{\cos(a+b)\Lambda}{a+b} \right] \\ &= \frac{1}{a} \delta(a-b) \quad \text{for } a > 0 \text{ and } b > 0, \end{aligned} \quad (\text{C.3})$$

where we used $J_l(0) = 0$ for $l > 0$ and the Hankel asymptotic form

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left[\cos \left(x - \frac{2\nu+1}{4} \pi \right) + \mathcal{O}(x^{-1}) \right] \quad \text{as } x \rightarrow \infty \quad (\text{C.4})$$

and the delta function defined by the weak limit¹⁷

$$\lim_{\Lambda \rightarrow \infty} \frac{\sin(\Lambda x)}{\pi x} \equiv \delta(x). \quad (\text{C.5})$$

Thus we obtain

$$(\psi_{m,s}, \psi_{m',s'}) = \frac{\pi}{2} \delta_{s,s'} \frac{|N_m|^2}{m} \delta(m - m'). \quad (\text{C.6})$$

If $|N_m| = \sqrt{\frac{2m}{\pi}}$, we get $(\psi_{m,s}, \psi_{m',s'}) = \delta_{s,s'} \delta(m - m')$.

Next we check the orthogonality between the massive eigenfunction and the massless eigenfunctions. For massless $m = 0$, we consider the $s \neq 0$ case and the $s = 0$ case separately.

The general solution of eq.(4.14) for $s \neq 0$ is

$$\psi_{0,s}(r, \theta) = (A r^{-|s|} + B r^{|s|}) e^{is\theta}, \quad (\text{C.7})$$

where A and B are constants. For simplicity, we consider the $s > 0$ case. The $s < 0$ case is also discussed similarly. The scalar product of $\psi_{0,s}$ and $\psi_{m,s'}$ for $s > 0$ is

$$\begin{aligned} (\psi_{0,s}, \psi_{m,s'}) &= 2\pi\delta_{s,s'} N_m \int_0^\infty dr r (A^* r^{-s} + B^* r^s) J_s(2mr) \\ &= 2\pi\delta_{s,s'} N_m \left[A^* \frac{m^{s-2}}{2(s-1)!} + B^* \sqrt{\frac{2}{\pi}} \lim_{\Lambda \rightarrow \infty} \frac{\Lambda^{s+\frac{1}{2}}}{(2m)^{\frac{3}{2}}} \cos \left(2m\Lambda - \frac{2s+3}{4} \pi \right) \right], \end{aligned} \quad (\text{C.8})$$

¹⁷Note that the delta function defined in this way makes sense only for the smooth function with compact support. In other word, it satisfy that $\int dx \lim_{\Lambda \rightarrow \infty} \frac{\sin(\Lambda x)}{\pi x} f(x) = \int dx \lim_{\Lambda \rightarrow \infty} \frac{e^{i\Lambda x} - e^{-i\Lambda x}}{2\pi i x} f(x) = f(0)$ for any smooth functions $f(x)$ with compact support, in the sense of distributions. In our case, we suppose that this delta function and the smooth function together will be integrated.

where we use the asymptotic form (C.4) and the following results for positive integer l

$$\begin{aligned} \int_0^y dx x^{l+1} J_l(x) &= y^{l+1} J_{l+1}(y) \\ \int_0^y dx x^{-l+1} J_l(x) &= -y^{-l+1} J_{l-1}(y) + \frac{1}{2^{l-1}(l-1)!}. \end{aligned} \quad (\text{C.9})$$

For $m \neq 0$, the second term of eq.(C.8) is zero because of the same reason as the definition of the delta function using *sinc* function. Thus the r.h.s. of eq.(C.8) vanishes only if $A = 0$. Therefore, the solution eq.(4.14) for $s > 0$ is $\psi_{0,s} = r^s e^{is\theta}$. In the same way, we find that the solution for $s < 0$ is $\psi_{0,s} = r^{-s} e^{is\theta}$.

The general solution of eq.(4.14) for $s = 0$ is

$$\psi_{0,0}(r, \theta) = A \log r + B, \quad (\text{C.10})$$

where A and B are constants. The scalar product of $\psi_{0,0}$ and $\psi_{m,s}$ is

$$\begin{aligned} (\psi_{0,0}, \psi_{m,s}) &= 2\pi \delta_{s,0} N_m \int_0^\infty dr r (A^* \log r + B^*) J_s(2mr) \\ &= 2\pi \delta_{s,s'} N_m \left[-\frac{A^*}{(2m)^2} + \lim_{\Lambda \rightarrow \infty} (A^* \log \Lambda + B^*) \sqrt{\frac{2}{\pi}} \frac{\Lambda^{\frac{1}{2}}}{(2m)^{\frac{3}{2}}} \cos \left(2m\Lambda - \frac{3}{4}\pi \right) \right], \end{aligned} \quad (\text{C.11})$$

where we use the asymptotic form (C.4), eq.(C.9) and the following result

$$\int_0^y dx x \log x J_0(x) = y J_1(y) + J_0(y) - 1. \quad (\text{C.12})$$

For $m \neq 0$, the second term of eq.(C.11) is zero again and the r.h.s. of eq.(C.11) vanishes only if $A = 0$. Therefore the solution of eq.(4.14) for $s = 0$ is a constant.

If we collect these three cases, we obtain

$$\psi_{0,s}(r, \theta) = B r^{|s|} e^{is\theta}. \quad (\text{C.13})$$

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